Solving Differential Equations in Finance and Economics
The Integrating Factor Technique - Part II

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If we are currently standing at time zero then asset price at time zero is known but asset price at time $t > 0$ is unknown (i.e. is a random variable). If at time zero we are given the range of possible asset prices at time $t > 0$ we want to be able to account for the components of the change in asset price given asset price at time t. The change in asset price is therefore a differential equation. In this white paper we will guess as to the form of the equation that defines the change in asset price and then develop the mechanics to prove that that equation is valid. To this end we will use the following hypothetical problem...

Our Hypothetical Problem

We are currently standing at time zero where stock price is $100. We will define the change in stock price over the time interval $[0, t]$ to consist of expected return (always positive), unexpected return (positive or negative) and dividends paid (always positive). The equation for the change in stock price is therefore...

$$\text{Change in stock price} = \text{Expected return} + \text{Unexpected return} - \text{Dividends paid} \tag{1}$$

Problem to solve: Free cash flow per share is expected to grow at a rate of 4% per annum and the risk-adjusted discount rate is 12% per annum. If stock price at the end of year three is $130 then account for the total change in stock price using Equation (1) above as your guide.

Equation for Stock Price

We will define the variable C_t to be annualized free cash flow at time t, the variable μ to be the expected cash flow growth rate, the variable σ to be growth rate volatility, and the variable W_t to be a Brownian motion with mean zero and variance equal to the passage of time. The stochastic differential equation (SDE) for the change in annualized free cash flow at time t is...

$$\delta C_t = \mu \, C_t \, \delta t + \sigma \, \delta W_t \;\; \text{...where...} \;\; \delta W_t \sim N\left[0, \delta t\right] \tag{2}$$

The solution to the SDE in Equation (2) above is the equation for free cash flow at time t. If we are currently at time zero then the equation for free cash flow at time t where λ is the random growth rate is...

$$C_t = C_0 \, \text{Exp}\left\{\lambda\right\} \;\; \text{...where...} \;\; \lambda = \left(\mu - \frac{1}{2}\,\sigma^2\right)t + \sigma \, W_t \;\; \text{...and...} \;\; W_t \sim N\left[0, t\right] \tag{3}$$

We will need a generalized equation for free cash flow at any time within the time interval $[0, t]$. Using Equation (3) above the equation for free cash flow at time $0 \le s \le t$ is...

$$C_s = C_0 \, \text{Exp}\left\{\frac{\lambda}{t}\, s\right\} \;\; \text{...where...} \;\; 0 \le s \le t \tag{4}$$

Using Equation (4) above the equation for expected annualized free cash flow at time $0 \le s \le t$ is...

$$\mathbb{E}\left[C_s\right] = C_0 \, \text{Exp}\left\{\mu \, s\right\} \;\; \text{...where...} \;\; 0 \le s \le t \tag{5}$$

We will define the variable V_t to be stock price at time t and the variable κ to be the risk-adjusted discount rate. Using Equation (5) above the equation for stock price is...

$$V_t = \int\limits_t^\infty C_t \,\mathrm{Exp}\left\{\mu\,(v-t)\right\}\mathrm{Exp}\left\{-\,\kappa\,(v-t)\right\}\delta v = \theta\, C_t \;\text{ ...where... }\; \theta = \frac{1}{\kappa-\mu} \;\text{ ...and... }\; \kappa > \mu \tag{6}$$

Note that stock price at time t per the equation above is based on annualized cash flow at time t, which is known at time t, and the expected cash flow growth rate μ, which is the expected go-forward growth rate and the actual cash flow growth rate over the time interval $[0, t]$ is irrelevant.

Defining the Differential Equation

We will define the variable R_s to be the expected return on investment (ROI) over the infinitesimally small time interval $[s, s+\delta s]$. Expected ROI is what we expect to earn on our investment (i.e. the risk-adjusted discount rate). Using Equation (6) above the equation for expected return on investment at time s is...

$$R_s = \kappa\, V_s\, \delta s \tag{7}$$

We will define the variable G_s to be the unexpected return on investment over the infinitesimally small time interval $[s, s+\delta s]$. Per Equation (5) above we expect cash flow to grow at rate μ but the actual cash flow growth rate is a random variable. Since per Equation (6) above asset value is a fixed multiple of actual annualized cash flow we will recognize an unexpected gain if the actual growth rate is greater than μ and and an unexpected loss if the actual growth rate is less than μ. The equation for unexpected return on investment at time s is...

$$G_s = \left(\frac{\lambda}{t} - \mu\right) V_s\, \delta s \tag{8}$$

We will define the variable D_s to be dividends paid over the infinitesimally small time interval $[s, s+\delta s]$. By definition all free cash flow is paid out as dividends so using Equation (4) above the equation for dividends paid at time s is...

$$D_s = C_s\, \delta s \tag{9}$$

Using Equation (1) as our guide and Equations (7), (8) and (9) above the equation for the total change in asset value is...

$$\begin{aligned} \delta V_s &= R_s + G_s - D_s \\ &= \kappa\, V_s\, \delta s + \left(\frac{\lambda}{t} - \mu\right) V_s\, \delta s - C_s\, \delta s \\ &= \left(\kappa + \frac{\lambda}{t} - \mu\right) V_s\, \delta s - C_s\, \delta s \end{aligned} \tag{10}$$

Note that we can rewrite Equation (10) above as the following differential equation...

$$\frac{\delta V_s}{\delta s} = \left(\kappa + \frac{\lambda}{t} - \mu\right) V_s - C_s \tag{11}$$

If we move all terms that are applicable to stock price to the left hand side of the equation then Equation (11) above becomes...

$$\frac{\delta V_s}{\delta s} - \left(\kappa + \frac{\lambda}{t} - \mu\right) V_s = -C_s \tag{12}$$

Solving the Differential Equation

To solve Equation (12) above we will use the Integrating Factor Technique for solving differential equations. We will start by defining the function I_s to be the intergrating factor whose equation is...

$$I_s = \mathrm{Exp}\left\{-\left(\kappa + \frac{\lambda}{t} - \mu\right) s\right\} \;\text{ ...where... }\; \frac{\delta I_s}{\delta s} = -\left(\kappa + \frac{\lambda}{t} - \mu\right) \mathrm{Exp}\left\{-\left(\kappa + \frac{\lambda}{t} - \mu\right) s\right\} \tag{13}$$

Using the definition of the integrating factor in Equation (13) above the equation for the derivative of the product of the integrating factor and stock price with respect to time is...

$$
\begin{aligned}
\frac{\delta\, I_s V_s}{\delta s} &= \frac{\delta I_s}{\delta s}\, V_s + \frac{\delta V_s}{\delta s}\, I_s \\
&= -\left(\kappa + \frac{\lambda}{t} - \mu\right) \mathrm{Exp}\left\{ -\left(\kappa + \frac{\lambda}{t} - \mu\right) s \right\} V_s + \frac{\delta V_s}{\delta s}\, \mathrm{Exp}\left\{ -\left(\kappa + \frac{\lambda}{t} - \mu\right) s \right\} \\
&= \mathrm{Exp}\left\{ -\left(\kappa + \frac{\lambda}{t} - \mu\right) s \right\} \left[\frac{\delta V_s}{\delta s} - \left(\kappa + \frac{\lambda}{t} - \mu\right) V_s \right] \qquad (14)
\end{aligned}
$$

If we multiply Equation (12) above by the integrating factor as defined by Equation (13) above then the equation for that product is...

$$
\mathrm{Exp}\left\{ -\left(\kappa + \frac{\lambda}{t} - \mu\right) s \right\} \left[\frac{\delta\, V_s}{\delta s} - \left(\kappa + \frac{\lambda}{t} - \mu\right) V_s \right] = -\mathrm{Exp}\left\{ -\left(\kappa + \frac{\lambda}{t} - \mu\right) s \right\} C_s \qquad (15)
$$

Using Equation (14) above we can rewrite Equation (15) above as...

$$
\frac{\delta\, I_s V_s}{\delta s} = -\mathrm{Exp}\left\{ -\left(\kappa + \frac{\lambda}{t} - \mu\right) s \right\} C_s \qquad (16)
$$

Using Equation (14) above we can rewrite Equation (15) above as...

$$
\begin{aligned}
\frac{\delta\, I_s V_s}{\delta s} &= -\mathrm{Exp}\left\{ -\left(\kappa + \frac{\lambda}{t} - \mu\right) s \right\} C_0\, \mathrm{Exp}\left\{ \frac{\lambda}{t}\, s \right\} \delta s \\
&= -\, C_0\, \mathrm{Exp}\left\{ -\left(\kappa - \mu\right) s \right\} \delta s \qquad (17)
\end{aligned}
$$

If we integrate both sides of Equation (17) above over the time interval $[0, t]$ then that equation becomes...

$$
\int_0^t \frac{\delta\, I_s V_s}{\delta s}\, \delta s = -\, C_0 \int_0^t \mathrm{Exp}\left\{ -\left(\kappa - \mu\right) s \right\} \delta s \qquad (18)
$$

Using Equation (6) above the solution to Equation (18) above is...

$$
\begin{aligned}
I_t V_t - I_0 V_0 &= \frac{C_0}{\kappa - \mu}\, \mathrm{Exp}\left\{ -\left(\kappa - \mu\right) s \right\} \Big[_0^t \\
&= \frac{C_0}{\kappa - \mu} \left(\mathrm{Exp}\left\{ -\left(\kappa - \mu\right) t \right\} - 1 \right) \\
&= \theta\, C_0\, \mathrm{Exp}\left\{ -\left(\kappa - \mu\right) t \right\} - V_0 \qquad (19)
\end{aligned}
$$

Using Equation (13) above the values of the integrating factor at time t and at time zero are...

$$
I_t = \mathrm{Exp}\left\{ -\left(\kappa + \frac{\lambda}{t} - \mu\right) t \right\} \text{ ...and... } I_0 = \mathrm{Exp}\left\{ -\left(\kappa + \frac{\lambda}{t} - \mu\right) \times 0 \right\} = 1 \qquad (20)
$$

Using Equation (20) above we can rewrite Equation (19) as...

$$
\begin{aligned}
\mathrm{Exp}\left\{ -\left(\kappa + \frac{\lambda}{t} - \mu\right) t \right\} V_t - V_0 &= \theta\, C_0\, \mathrm{Exp}\left\{ -\left(\kappa - \mu\right) t \right\} - V_0 \\
\mathrm{Exp}\left\{ -\left(\kappa + \frac{\lambda}{t} - \mu\right) t \right\} V_t &= \theta\, C_0\, \mathrm{Exp}\left\{ -\left(\kappa - \mu\right) t \right\} \\
V_t &= \theta\, C_0\, \mathrm{Exp}\left\{ \frac{\lambda}{t}\, t \right\} \\
V_t &= \theta\, C_0\, \mathrm{Exp}\left\{ \lambda \right\} \qquad (21)
\end{aligned}
$$

Using Equation (4) above the solution to Equation (21) above is...

$$V_t = \theta\, C_t \tag{22}$$

Per Equation (22) above the solution to our differential equation is stock price at time t per Equation (6) above such that...

$$\delta V_s = R_s + G_s - D_s \;\text{ ...where... }\; V_s = \theta\, C_s \tag{23}$$

Hypothetical Problem Solution

To answer the hypothetical problem our first step will be to solve for the random cash flow growth rate λ. Using Equations (3) and (6) above the equation for cash flow growth rate is...

$$\text{if... } V_t = \theta\, C_0 \,\mathrm{Exp}\left\{\lambda\right\} \text{ ...then... } \lambda = \ln\left(\frac{V_t}{V_0}\right) \text{ ...such that... } \lambda = \ln\left(\frac{130}{100}\right) = 0.2624 \tag{24}$$

The next step is to solve for the cash flow valuation multiple θ. Using Equation (6) above the equation for the valuation multiple is...

$$\theta = \frac{1}{\kappa - \mu} = \frac{1}{0.12 - 0.04} = 12.50 \tag{25}$$

The next step is to solve for annualized cash flow at time zero. Using Equation (6) above the equation for annualized cash flow is...

$$V_0 = \theta\, C_0 \text{ ...such that... } C_0 = \frac{V_0}{\theta} = \frac{100}{12.50} = 8.00 \tag{26}$$

We will define the variable $\bar{R}_t$ to be total expected return on investment over the time interval $[0, t]$. Using Equations (4) and (7) above and Appendix Equation (30) below the equation for total expected return on investment is...

$$\bar{R}_t = \int\limits_0^t R_s = \int\limits_0^t \kappa\, V_s\, \delta s = \kappa\, \theta\, C_0 \int\limits_0^t \mathrm{Exp}\left\{\frac{\lambda}{t}\, s\right\} \delta s = \frac{\kappa\, \theta\, t}{\lambda}\, C_0 \Big(\mathrm{Exp}\left\{\lambda\right\} - 1\Big) \tag{27}$$

We will define the variable $\bar{G}_t$ to be total unexpected return on investment over the time interval $[0, t]$. Using Equations (4) and (8) above and Appendix Equation (30) below the equation for total unexpected return on investment is...

$$\bar{G}_t = \int\limits_0^t G_s = \int\limits_0^t \left(\frac{\lambda}{t} - \mu\right) V_s\, \delta s = \frac{\lambda - \mu\, t}{t}\, \theta\, C_0 \int\limits_0^t \mathrm{Exp}\left\{\frac{\lambda}{t}\, s\right\} \delta s = \frac{\theta(\lambda - \mu\, t)}{\lambda}\, C_0 \Big(\mathrm{Exp}\left\{\lambda\right\} - 1\Big) \tag{28}$$

We will define the variable $\bar{D}_t$ to be total dividends paid over the time interval $[0, t]$. Using Equations (4) and (9) above and Appendix Equation (30) below the equation for total dividends paid is...

$$\bar{D}_t = \int\limits_0^t D_s = \int\limits_0^t C_s\, \delta s = C_0 \int\limits_0^t \mathrm{Exp}\left\{\frac{\lambda}{t}\, s\right\} \delta s = \frac{t}{\lambda}\, C_0 \Big(\mathrm{Exp}\left\{\lambda\right\} - 1\Big) \tag{29}$$

Answer: Using Equations (24) to (29) above the components of the change in stock price from period zero to period four are...

Begin stock price	100.00	
Expected ROI	54.89	Use Equation (27) above
Unexpected ROI	11.70	Use Equation (28) above
Dividends paid	-36.59	Use Equation (29) above
End stock price	130.00	

Appendix

A. The solution to the following integral is...

$$\int\limits_0^t \mathrm{Exp}\left\{\frac{\lambda}{t}\, s\right\} \delta s = \frac{t}{\lambda}\, \mathrm{Exp}\left\{\frac{\lambda}{t}\, s\right\} \Big[_0^t = \frac{t}{\lambda}\Big(\mathrm{Exp}\left\{\lambda\right\} - 1\Big) \tag{30}$$